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Hardness of computing quantum invariants of 3-manifolds with restricted topology

jcgeo25 : Jeunes Chercheuses et Chercheurs en Géométrie

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Quantum invariants

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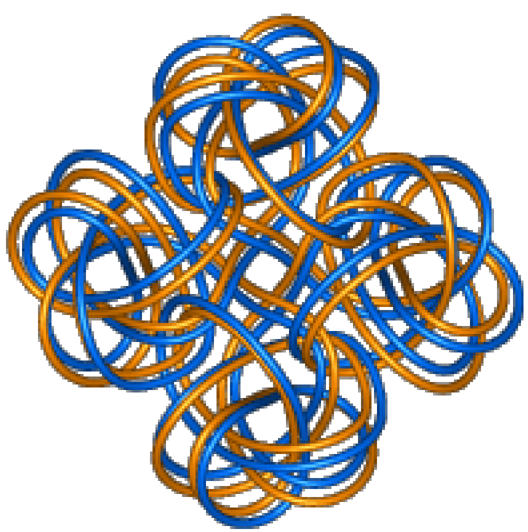

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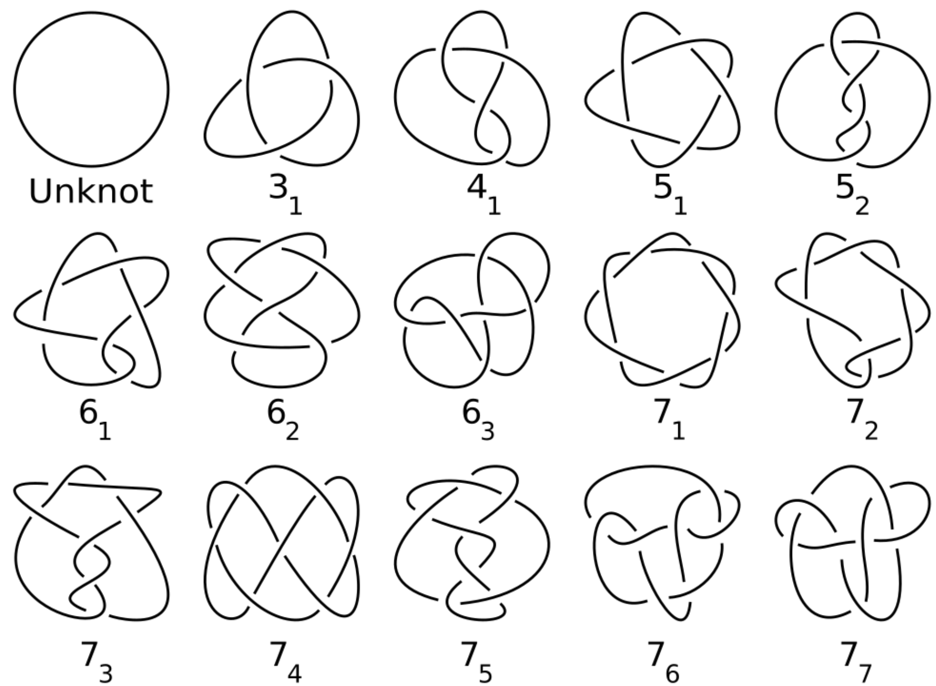

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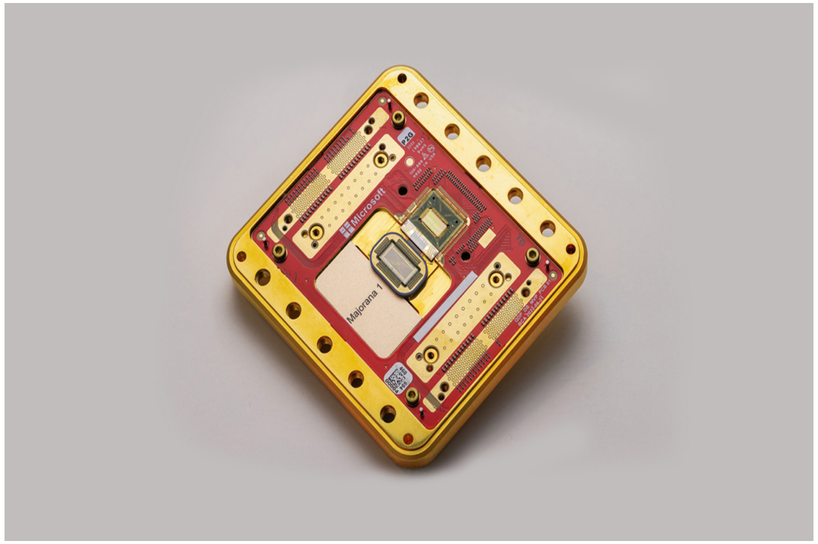
- used in **quantum computing**



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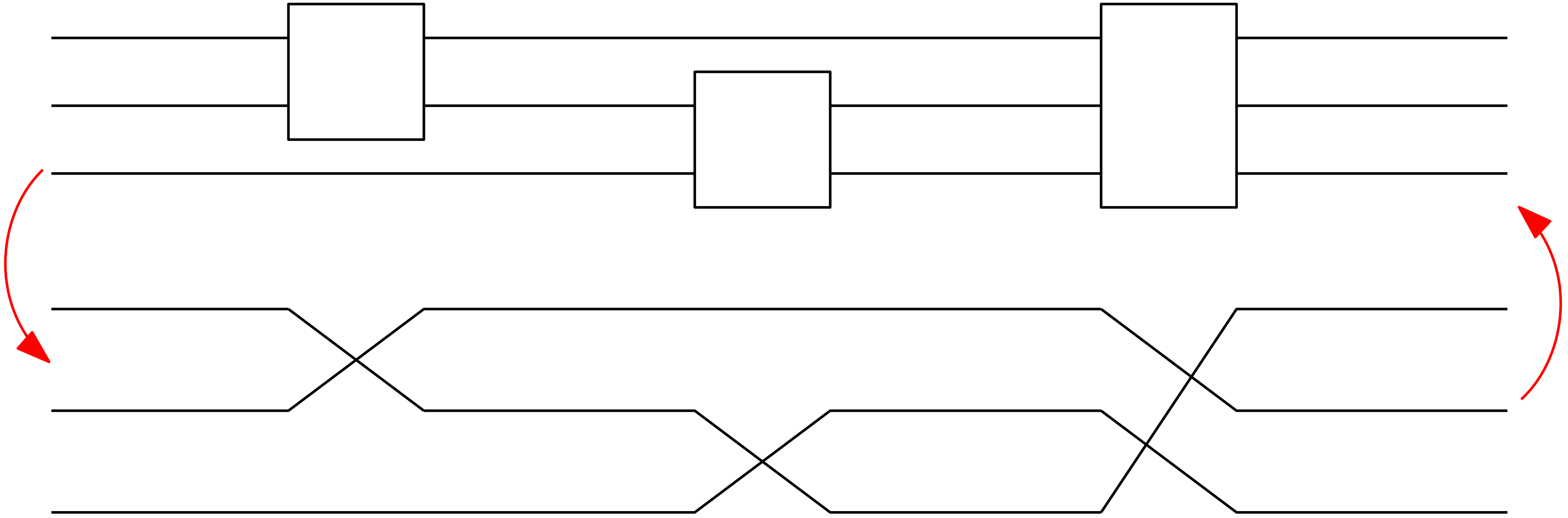

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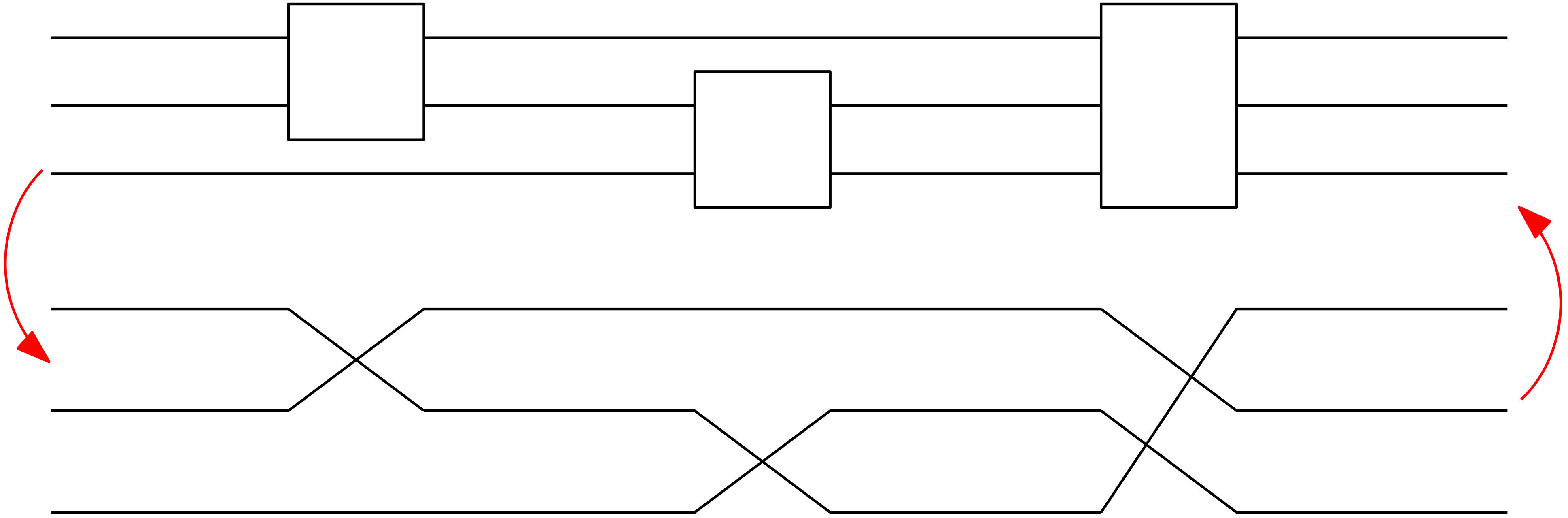

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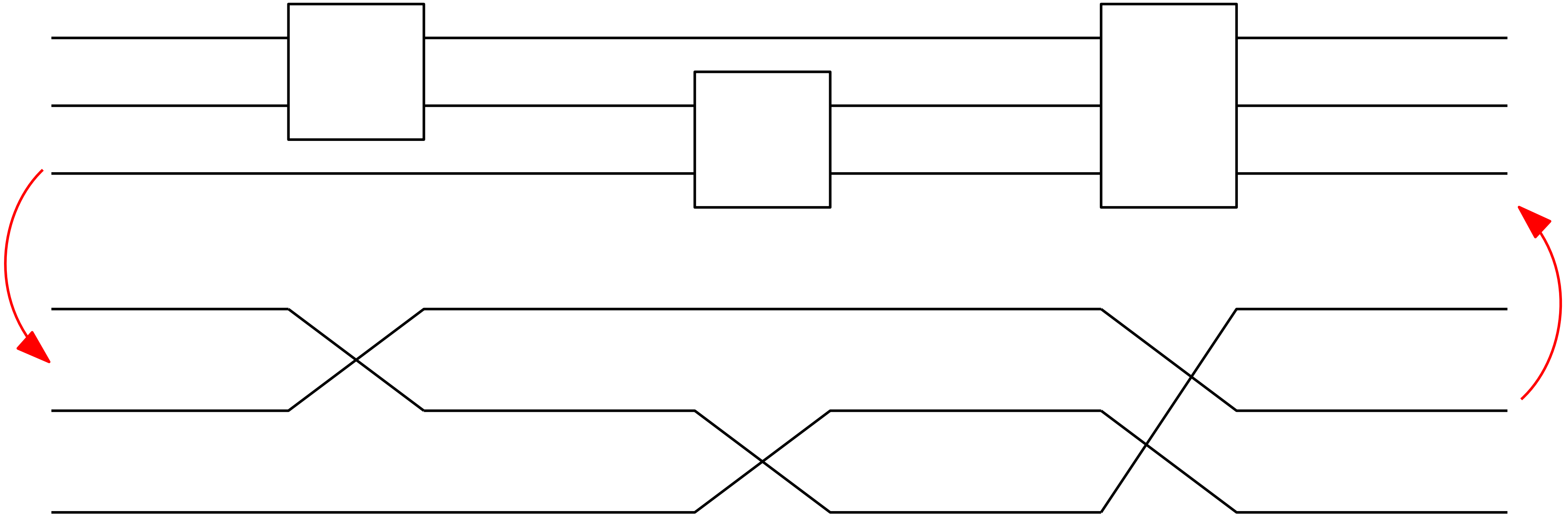

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Theorem (Kuperberg, 2009; Algic and Lo, 2014):
Exact computations (or even good approximations) of the RT invariant for some choices of $\mathcal{C}$ (e.g., Jones polynomial for links and WRT for closed 3-manifolds) are #P-hard

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Consider a logic term with free variables

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Theorem (Aaronson, 2005):
Computing the probability of a quantum circuit giving TRUE is #P-hard

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Does restricting the topology yields to easier algorithms?

- A manifold M is **irreducible*** if M is not homemorphic to the direct sum $N_1 \# N_2$ where $N_1, N_2 \neq S^3$
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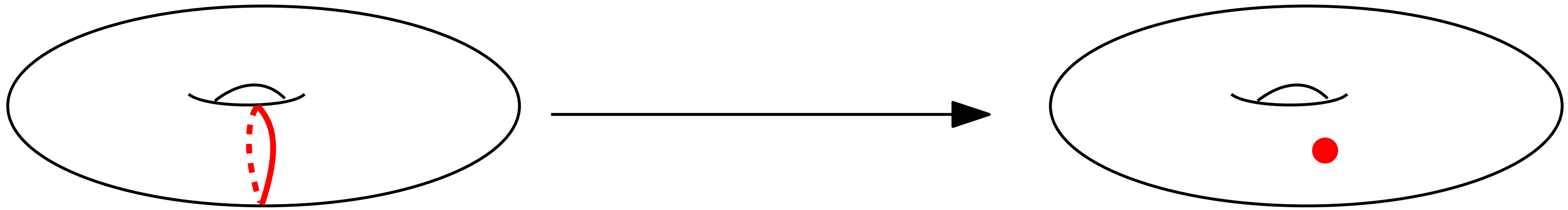

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(H.E. and C.M., 2025) No

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$$M \xrightarrow{\text{restrict the topology}} M' \xrightarrow{\mathcal{M}'} \langle M \rangle_{\mathcal{C}}$$

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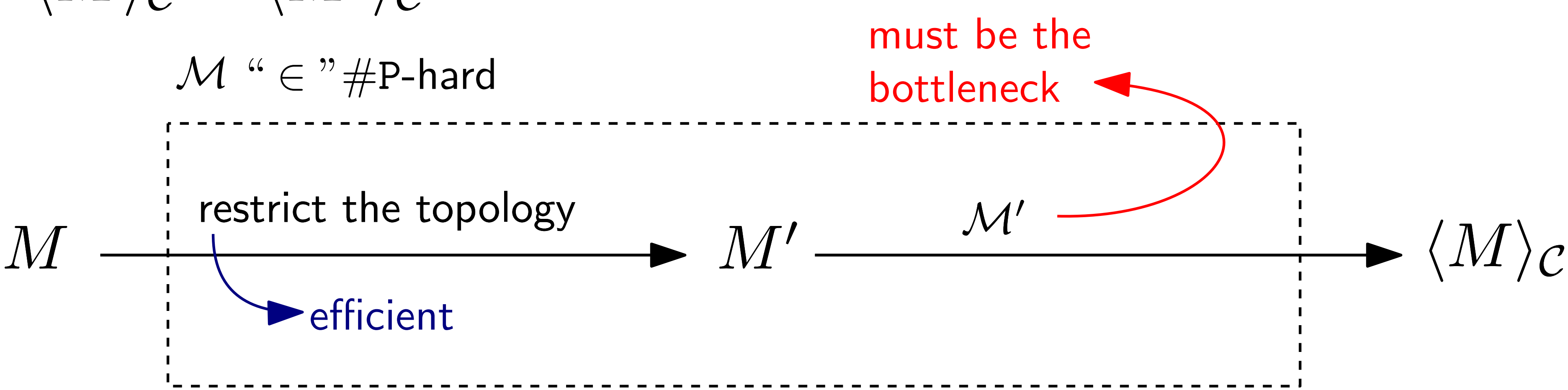

Thank you!